

JUMP LOCI FOR THE RANK OF MATRICES AND BETTI NUMBERS OF CHAIN COMPLEXES OVER LAURENT POLYNOMIAL RINGS

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ABSTRACT. Let \mathcal{K} be a non-empty set of ideals of the commutative ring R , closed under taking smaller ideals. A subset X of the group ring $R[\mathbb{Z}^s]$ is called a \mathcal{K} -set if the ideal generated by the coefficients of the elements of X is in \mathcal{K} . For X not a \mathcal{K} -set we investigate the set of those $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ such that $p_*(X)$ is a \mathcal{K} -set. We also consider corresponding notions of rank of matrices and BETTI numbers of chain complexes; this includes an analysis of the case of MCCOY rank. Our setup also recovers results on jump loci obtained by KOHNO and PAJITNOV.

INTRODUCTION

Let R be a commutative integral domain with unit. A group homomorphism $p: \mathbb{Z}^s \longrightarrow \mathbb{Z}^t$ determines a ring homomorphism $p_*: R[\mathbb{Z}^s] \longrightarrow R[\mathbb{Z}^t]$. Given a bounded chain complex C of finitely generated free $R[\mathbb{Z}^s]$ -modules we obtain the induced chain complex $p_*(C) = C \otimes_{R[\mathbb{Z}^s]} R[\mathbb{Z}^t]$ of finitely generated free $R[\mathbb{Z}^t]$ -modules. Over the ring R we have a meaningful notion of “rank” for matrices and free modules, and can thus define BETTI numbers $b_k(C)$ and $b_k(p_*(C))$ of our chain complexes. KOHNO and PAJITNOV proved the following result characterising the jump loci of the BETTI numbers with respect to varying the group homomorphism p :

Theorem 0.1 (KOHNO and PAJITNOV [KP14, Theorem 7.3]). *Let $k \in \mathbb{Z}$ and $q \geq 0$ be given. There exists a finite family of proper direct summands $G_i \subset \text{hom}(\mathbb{Z}^s, \mathbb{Z})$ such that the inequality $b_k(C) + q < b_k(p_*(C))$ holds if and only if there is an index i with $p \in G_i^t \subseteq \text{hom}(\mathbb{Z}^s, \mathbb{Z})^t = \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$.*

The first step of the proof is to characterise those $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ which satisfy $p_*(\Delta) = 0$, for a fixed non-zero $\Delta \in R[\mathbb{Z}^s]$. Next one establishes a variant of the theorem for jump loci of the rank of matrices, characterising the condition $\text{rank}(p_*(A)) < \text{rank}(A)$ for a fixed matrix A with entries in $R[\mathbb{Z}^s]$. Finally the actual theorem can be verified by considering the ranks of the differentials in the chain complex.

In the present paper, we take the jump from the annihilation condition $p_*(\Delta) = 0$ to a more inclusive formulation. Let \mathcal{K} be a non-empty set of

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ideals of R such that¹ if $I \in \mathcal{K}$ and J is an ideal contained in I , then $J \in \mathcal{K}$. With respect to \mathcal{K} and Δ , we formulate the following condition on p : *The ideal of R generated by the coefficients of the element $p_*(\Delta) \in R[\mathbb{Z}^t]$ lies in \mathcal{K} .* This should be thought of as saying that $p_*(\Delta)$ satisfies a certain property encoded by \mathcal{K} . If \mathcal{K} consists of the zero ideal only, the property is that of being zero.

With respect to the “property” \mathcal{K} we establish a notion of \mathcal{K} -rank of matrices and \mathcal{K} -BETTI numbers for chain complexes of finitely generated free modules, and characterise their jump loci. We allow R to be an arbitrary commutative ring which may even be non-unital. It is surprising that the aforementioned results can be established in this generality. The price to pay is that one cannot expect to have *proper* direct summands G_i any more.

Among the many possible sets \mathcal{K} two deserve special mention. We assume a unital commutative ring R for now. If we take \mathcal{K} to consist of the zero ideal only, we recover the original result of KOHNO and PAJITNOV [KP14]. If we take \mathcal{K} to be the set of all ideals having non-trivial annihilator then the \mathcal{K} -rank of a matrix (Definition 4.1) is precisely the MCCOY-rank of a matrix as introduced in [McC42, §2].

Notation and conventions. Throughout the paper, R denotes a fixed commutative ring, possibly non-unital. We will concern ourselves with the group rings $R[\mathbb{Z}^s]$ and $R[\mathbb{Z}^t]$; their elements are written in the form $\sum_{a \in \mathbb{Z}^s} r_a x^a$ and $\sum_{b \in \mathbb{Z}^t} \rho_b y^b$, respectively, with almost all of the coefficients r_a and ρ_b being zero. We let $\mathbb{Z}^{s*} = \text{hom}(\mathbb{Z}^s, \mathbb{Z})$ stand for the \mathbb{Z} -dual of \mathbb{Z}^s . The group $\text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ is often identified with $(\mathbb{Z}^{s*})^t$. Any homomorphism $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ induces an R -algebra homomorphism $p_*: R[\mathbb{Z}^s] \longrightarrow R[\mathbb{Z}^t]$, which maps $\Delta = \sum_{a \in \mathbb{Z}^s} r_a x^a \in R[\mathbb{Z}^s]$ to

$$p_*(\Delta) = \sum_{b \in \mathbb{Z}^t} \left(\sum_{a \in p^{-1}(b)} r_a \right) \cdot y^b \in R[\mathbb{Z}^t]. \quad (0.2)$$

The map $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ also induces a map of sets of matrices

$$p_*: \mathcal{M}_{m,n}(R[\mathbb{Z}^s]) \longrightarrow \mathcal{M}_{m,n}(R[\mathbb{Z}^t])$$

by applying the ring homomorphism p_* to each matrix element.

1. PROPERTIES OF IDEALS AND MODULES

Suppose that \mathcal{K} is a set of ideals of R . This set encodes a “property” that elements of the group ring $R[\mathbb{Z}^n]$ may or may not possess. The simplest case is that \mathcal{K} consists of the zero ideal only, in which case the property in question is “being 0”.

Definition 1.1. (1) An ideal I of R is called a \mathcal{K} -ideal if $I \in \mathcal{K}$.

(2) A subset $X \subseteq R$ is called a \mathcal{K} -set if the ideal $\langle X \rangle$ generated by X is a \mathcal{K} -ideal.

¹Such a set \mathcal{K} is usually called an order ideal in the partially ordered set of ideals of R , but this terminology seems less than ideal in the present context.

Definition 1.2. Let X be subset of the group ring $R[\mathbb{Z}^n]$.

- (1) We let iX denote the ideal generated by the set of coefficients of the elements of X .
- (2) We say that X is a \mathcal{K} -set provided that iX is a \mathcal{K} -ideal, i.e., provided that the set of coefficients of elements of X is a \mathcal{K} -set.
- (3) An R -submodule M of $R[\mathbb{Z}^n]$ is called a \mathcal{K} -module provided it is a \mathcal{K} -set.

In general, we will only be interested in sets \mathcal{K} which are closed under taking smaller ideals:

Definition 1.3. We call \mathcal{K} *hereditary* if \mathcal{K} is non-empty, and if $I \in \mathcal{K}$ and $J \subseteq I$ together imply $J \in \mathcal{K}$, for any $J \trianglelefteq R$.

As mentioned before, two relevant hereditary sets are

$$\mathcal{K}_0 = \{\{0\}\} \quad \text{and} \quad \mathcal{K}_1 = \{I \trianglelefteq R \mid \text{ann}_R(I) \neq \{0\}\} ;$$

the former corresponds to vanishing conditions, the latter relates to the MCCOY-rank of matrices as explained in §4 below. If V is a fixed injective R -module we have the hereditary set

$$\mathcal{K}_{E,V} = \{I \trianglelefteq R \mid E(I) \text{ embeds into } V\} ,$$

where $E(I)$ denotes an injective hull of the R -module I ; more generally, if \mathcal{V} is a family of injective modules, we obtain a hereditary set

$$\mathcal{K}_{E,\mathcal{V}} = \{I \trianglelefteq R \mid E(I) \text{ embeds into } V \text{ for some } V \in \mathcal{V}\} .$$

Any union of hereditary sets is hereditary, and in fact $\mathcal{K}_{E,\mathcal{V}} = \bigcup_{V \in \mathcal{V}} \mathcal{K}_{E,V}$. Any subset $X \subseteq R$ containing 0 gives rise to the hereditary set $\mathcal{K}_{\subseteq X} = \{I \trianglelefteq R \mid I \subseteq X\}$, and, if X contains at least one element other than 0, the hereditary set $\mathcal{K}_{\subsetneq X} = \{I \trianglelefteq R \mid I \subsetneq X\}$. In particular, we can consider the hereditary set $\mathcal{K}_{\subsetneq J(R)}$ in case the JACOBSON radical $J(R)$ of R is non-trivial. For a unital ring R , this is the hereditary set of all superfluous ideals different from $J(R)$, where I is *superfluous* if $I + J = R$ implies $J = R$, for all $J \trianglelefteq R$.

Before we construct more examples of hereditary sets, we prove a basic result relating the property of “being a \mathcal{K} -set” for a set $X \subseteq R[\mathbb{Z}^s]$ and its image $p_*(X)$, for $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. Those p where $p_*(X)$ is a \mathcal{K} -set form the “jump loci” from the title of the paper.

Lemma 1.4. *Let \mathcal{K} be a hereditary set of ideals of R , and let $X \subseteq R[\mathbb{Z}^s]$ be a subset. For any $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$, if X is a \mathcal{K} -set then so is $p_*(X)$.*

Proof. Let I be the ideal generated by the coefficients of elements of X ; we have $I \in \mathcal{K}$ by hypothesis on X . Let J be the ideal generated by the coefficients of elements of $p_*(X)$. The elements of $p_*(X)$ are of the form $p_*(\Delta)$, for $\Delta \in X$, and by formula (0.2) the coefficients of $p_*(\Delta)$ are sums of coefficients of Δ . But the latter coefficients are elements of I , hence so are the former. That is, the generators of J are elements of I whence $J \subseteq I$. As \mathcal{K} is hereditary we conclude $J \in \mathcal{K}$ so that $p_*(X)$ is a \mathcal{K} -set as claimed. \square

As our set \mathcal{K} always contains the zero ideal, we also have the following:

Lemma 1.5. *If X is a subset of the augmentation ideal $I_s = \ker 0_*$ of \mathbb{Z}^s , then $0_*(X)$ is a \mathcal{K} -set.* \square

Remark 1.6. If the hereditary set \mathcal{K} contains a unique maximal element J , then the conditions “being a \mathcal{K} -module” can be transformed into the annihilation condition “being trivial” by replacing the ground ring R with R/J .

Hereditary sets and filters of ideals. A *filter* of ideals is a non-empty set \mathcal{F} of ideals of R such that $J \in \mathcal{F}$ and $I \supseteq J$ together imply $I \in \mathcal{F}$. There is an intimate connection between hereditary sets of ideals and filters of ideals:

Proposition 1.7. (1) *Every filter \mathcal{F} of ideals determines a hereditary set of ideals given by $\mathcal{F}' = \{I \trianglelefteq R \mid \text{ann}_R(I) \in \mathcal{F}\}$, and the assignment $\mathcal{F} \mapsto \mathcal{F}'$ is inclusion-reversing.*
 (2) *Every hereditary set \mathcal{K} of ideals determines a filter of ideals given by $\mathcal{K}' = \{J \trianglelefteq R \mid \text{ann}_R(J) \in \mathcal{K}\}$, and the assignment $\mathcal{K} \mapsto \mathcal{K}'$ is inclusion-reversing.* \square

Hereditary sets of ideals can be constructed with the aid of Proposition 1.7, for example from the filter $\mathcal{F}_{\text{ess}} = \{J \trianglelefteq R \mid J \text{ is an essential ideal}\}$, where J is *essential* if $J \cap I = \{0\}$ implies $I = \{0\}$, for all $I \trianglelefteq R$. Using Proposition 1.7 twice, any hereditary set \mathcal{K} gives rise to another hereditary set $\mathcal{K}'' \supseteq \mathcal{K}$; specifically, $\mathcal{K}'_0 \subseteq \mathcal{F}_{\text{ess}}$ and hence $\mathcal{K}''_0 \supseteq \mathcal{F}'_{\text{ess}}$.

Any subset X of R defines a filter $\mathcal{F}_{\supseteq X} = \{J \trianglelefteq R \mid J \supseteq X\}$, and, if $X \neq R$, also the filter $\mathcal{F}_{\not\supseteq X} = \{J \trianglelefteq R \mid J \not\supseteq X\}$. Finally, we observe that any intersection of filters of ideals is again a filter.

2. PARTITION SUBGROUPS OF $\text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$

Definition 2.1. Suppose $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ is a partition² of a subset of \mathbb{Z}^s , so that $\pi = \coprod_{j=1}^k \pi_j \subseteq \mathbb{Z}^s$. To π we associate the ABELIAN group

$$H(\pi) = \{p \in \mathbb{Z}^{s*} \mid \forall j = 1, 2, \dots, k : p|_{\pi_j} \text{ is constant}\}$$

called the *partition subgroup associated to π* . For a set P of partitions of (possibly distinct) subsets of \mathbb{Z}^s , we define

$$H(P) = \bigcap_{\pi \in P} H(\pi),$$

and call $H(P)$ the *partition subgroup associated to P* .

Lemma 2.2. *Suppose P is a set of partitions of subsets of \mathbb{Z}^s .*

- (1) *If at least one part π_j of at least one partition $\pi \in P$ has at least two elements, then $H(P)$ is a proper subgroup of \mathbb{Z}^{s*} .*
- (2) *The subgroup $H(P)$ is contained in \mathbb{Z}^{s*} as a direct summand.*

Proof. Part (1) is trivial. Let us prove (2). In view of the canonical short exact sequence

$$0 \longrightarrow H(P) \xrightarrow{\subseteq} \mathbb{Z}^{s*} \longrightarrow \mathbb{Z}^{s*}/H(P) \longrightarrow 0$$

²We have $k = 0$ if and only if $\pi = \emptyset$, by convention.

it is enough to show that $\mathbb{Z}^{s*}/H(P)$ is torsion-free. For then $\mathbb{Z}^{s*}/H(P)$ is free ABELIAN, whence the short exact sequence splits.

So let $q \in \mathbb{Z}^{s*}$ be such that $[q] \in \mathbb{Z}^{s*}/H(P)$ is torsion. Then there exists a natural number $n \geq 1$ with

$$[nq] = n \cdot [q] = 0 \in \mathbb{Z}^{s*}/H(P) ,$$

that is, $nq \in H(P)$. By definition of $H(P)$ this means that the homomorphism nq is constant on each part π_j of each partition $\pi \in P$, which implies that q has the same property. Consequently, $q \in H(P)$ and thus $[q] = 0$. \square

Let $\Delta = \{\Delta^{(1)}, \Delta^{(2)}, \dots, \Delta^{(d)}\}$, for $d \geq 0$, be a finite subset of $R[\mathbb{Z}^s]$. To fix notation, we write

$$\Delta^{(i)} = \sum_{a \in \mathbb{Z}^s} r_a^{(i)} x^a \quad (\text{for } 1 \leq i \leq d).$$

Let $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ be a partition of $\text{supp}(\Delta) = \bigcup_{i=1}^d \text{supp}(\Delta^{(i)})$. We define ring elements

$$\Delta_j^{(i)} = \sum_{a \in \pi_j} r_a^{(i)} \in R \quad (\text{for } 1 \leq j \leq k \text{ and } 1 \leq i \leq d), \quad (2.3)$$

and denote the ideal generated by these elements by

$$\Delta_\pi = \langle \Delta_j^{(i)} \mid 1 \leq j \leq k, 1 \leq i \leq d \rangle \subseteq R .$$

Definition 2.4. Let \mathcal{K} be a hereditary set of ideals of R , cf. Definition 1.3. A \mathcal{K} -partition is a partition π of $\text{supp}(\Delta)$ such that Δ_π is a \mathcal{K} -ideal.

Lemma 2.5. Let \mathcal{K} be a hereditary set of ideals of R . Suppose that $i\Delta \notin \mathcal{K}$, that is, $i\Delta$ is not a \mathcal{K} -ideal. Suppose P is a set of partitions of subsets of \mathbb{Z}^s which contains at least one \mathcal{K} -partition π of $\text{supp}(\Delta)$. Then $H(P)$ is a proper subgroup of \mathbb{Z}^{s*} .

Proof. Since $i\Delta \notin \mathcal{K}$ we have $i\Delta \neq \{0\}$, so the set Δ contains a non-zero element whence $\text{supp}(\Delta) \neq \emptyset$. Let $\tau = (\tau_1, \tau_2, \dots, \tau_k) \in P$ be any partition of $\text{supp}(\Delta)$. If all parts τ_j of τ are singletons then the generators of Δ_τ , as specified in (2.3), are precisely the generators of $i\Delta$ so that $\Delta_\tau = i\Delta \notin \mathcal{K}$. That is, such a τ is not a \mathcal{K} -partition. But the stipulated partition π is a \mathcal{K} -partition; it follows that at least one part of π must have at least two elements. Now Lemma 2.2 applies, assuring us that $H(P)$ is a proper direct summand of \mathbb{Z}^{s*} . \square

3. JUMP LOCI FOR MODULES

In this section, \mathcal{K} denotes a fixed hereditary set of ideals of R in the sense of Definition 1.3.

Proposition 3.1. Let M be a non-trivial, finitely generated R -submodule of the group ring $R[\mathbb{Z}^s]$. There exist partition subgroups $G_1, G_2, \dots, G_\ell \subseteq \mathbb{Z}^{s*}$, where $\ell \geq 0$, such that for every $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$,

$$p_*(M) \text{ is a } \mathcal{K}\text{-module} \quad \Longleftrightarrow \quad p \in \bigcup_{j=1}^{\ell} G_j^t .$$

Here $G_j^t = \bigoplus_1^t G_j \subseteq (\mathbb{Z}^{s*})^t = \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. — More precisely, writing

$$\Delta = \{\Delta^{(1)}, \Delta^{(2)}, \dots, \Delta^{(d)}\} \subseteq M$$

for a finite generating set of the R -module M , the number ℓ is the number of \mathcal{K} -partitions of $\text{supp}(\Delta)$, and the groups G_j are the partition subgroups $H(\pi)$ associated to these partitions.

If M is not a \mathcal{K} -module then the groups G_j are proper subgroups of \mathbb{Z}^{s*} so that $\bigcup_{j=1}^\ell G_j^t \neq \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. We have $\ell > 0$ if and only if $0_*(M)$ is a \mathcal{K} -module, which certainly is the case if M is contained in the augmentation ideal of $R[\mathbb{Z}^s]$.

For ease of reading we delegate the main step of the proof to the following Lemma. Write $p = (p_1, p_2, \dots, p_t)$, with each component $p_i: \mathbb{Z}^s \longrightarrow \mathbb{Z}$ being an element of \mathbb{Z}^{s*} .

Lemma 3.2. *Let M and Δ be as in Proposition 3.1. The R -submodule $p_*(M)$ of $R[\mathbb{Z}^t]$ is a \mathcal{K} -module if and only if there exists a \mathcal{K} -partition π of $\text{supp}(\Delta)$ such that p is constant on each part π_j of π , that is, such that $p \in H(\pi)^t$.*

Proof. Let us now prove the “if” direction. Suppose that there is a \mathcal{K} -partition π of $\text{supp}(\Delta)$ with $p \in H(\pi)^t$; this last condition is equivalent to p being constant on each part π_j of π . As $p_*(M)$ is generated (as an R -module) by the set $p_*(\Delta)$, the ideal $ip_*(M)$ of R equals $ip_*(\Delta)$. We write

$$p_*(\Delta^{(i)}) = \sum_{b \in \mathbb{Z}^t} \rho_b^{(i)} y^b = \sum_{b \in \mathbb{Z}^t} \left(\sum_{a \in p^{-1}(b)} r_a^{(i)} \right) \cdot y^b$$

where $\rho_b^{(i)} = \sum_{a \in p^{-1}(b)} r_a^{(i)}$. As π is a partition of $\text{supp}(\Delta)$, and as p is constant on each part of π , we have the equality

$$\rho_b^{(i)} = \sum_j \sum_{\substack{a \in \pi_j \\ a \in p^{-1}(b)}} r_a^{(i)} = \sum_{\substack{j \\ p|_{\pi_j} \equiv a}} \sum_{a \in \pi_j} r_a^{(i)} = \sum_{\substack{j \\ p|_{\pi_j} \equiv a}} \Delta_j^{(i)}. \quad (3.3)$$

As π is a \mathcal{K} -partition, the ideal $\Delta_\pi = \langle \Delta_j^{(i)} \mid 1 \leq j \leq k, 1 \leq i \leq d \rangle$ is an element of \mathcal{K} . It contains all the $\Delta_j^{(i)}$ and hence all the $\rho_b^{(i)}$, by (3.3). As \mathcal{K} is hereditary the ideal generated by the $\rho_b^{(i)}$ thus also lies in \mathcal{K} . But this ideal is precisely $ip_*(M)$, as observed above, so $p_*(M)$ is a \mathcal{K} -module as desired.

To show the reverse implication suppose that $p_*(M)$ is a \mathcal{K} -module. Written more explicitly (using the notation from the previous paragraph) this means that the ideal $ip_*(M) = ip_*(\Delta)$ generated by the elements $\rho_b^{(i)} = \sum_{a \in p^{-1}(b)} r_a$, for $b \in \mathbb{Z}^t$ and $1 \leq i \leq d$, lies in \mathcal{K} . The requisite partition π of $\text{supp}(\Delta)$ is defined by declaring those intersections $\text{supp}(\Delta) \cap p^{-1}(b)$ which are non-empty to be the parts of π , where b varies over all of \mathbb{Z}^t . By construction, each component p_i of p is constant on each part of π ; on the part corresponding to $b = (b_1, b_2, \dots, b_t) \in \mathbb{Z}^t$, the component p_i takes the constant value b_i . The corresponding elements $\Delta_j^{(i)}$ are exactly the elements of the form $\rho_b^{(i)}$, so $\Delta_\pi = ip_*(\Delta) = ip_*(M)$ is a \mathcal{K} -ideal. We have thus shown that π is a \mathcal{K} -partition and $p \in H(\pi)^t$, as required. \square

Proof of Proposition 3.1. We re-state the conclusion of Lemma 3.2:

$$p_*(M) \text{ is a } \mathcal{K}\text{-module} \quad \Longleftrightarrow \quad p \in \bigcup_{\pi} H(\pi)^t,$$

the union extending over the finite set of all \mathcal{K} -partitions π of $\text{supp}(\Delta)$. Up to renaming the groups occurring on the right-hand side, this is the condition stated in the Proposition. We have verified that each $H(\pi)$ is a direct summand of \mathbb{Z}^{s*} in Lemma 2.2 above.

In case M is not a \mathcal{K} -module we know from Lemma 2.5 that the partition subgroups $H(\pi)$ are proper subgroups of \mathbb{Z}^{s*} . We have $\ell > 0$ if and only if $0 \in \bigcup_{j=1}^{\ell} G_j^t$ if and only if $0_*(M)$ is a \mathcal{K} -module. If M is contained in the augmentation ideal $I_s = \ker(0_*)$ of $R[\mathbb{Z}^s]$ then $0_*(M) = \{0\}$ is a \mathcal{K} -module so that the union $\bigcup_{j=1}^{\ell} G_j^t$ must contain 0; this forces $\ell > 0$. This finishes the proof of Proposition 3.1. \square

Corollary 3.4. *Let M_1, M_2, \dots, M_k be finitely generated R -submodules of $R[\mathbb{Z}^s]$. There are partition subgroups $G_1, G_2, \dots, G_{\ell} \subseteq \mathbb{Z}^{s*}$, where $\ell \geq 0$, such that for all $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$,*

$$\forall i = 1, 2, \dots, k : p_*(M_i) \text{ is a } \mathcal{K}\text{-module} \quad \Longleftrightarrow \quad p \in \bigcup_{j=1}^{\ell} G_j^t.$$

If at least one of the modules M_j is not a \mathcal{K} -module the groups G_j are proper subgroups of \mathbb{Z}^{s} so that $\bigcup_{j=1}^{\ell} G_j^t \neq \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. Moreover, $\ell > 0$ if and only if all the modules $0_*(M_j)$, for $1 \leq j \leq k$, are \mathcal{K} -modules, which is certainly the case if all of the modules M_j are contained in the augmentation ideal I_s of $R[\mathbb{Z}^s]$.*

Proof. Let $\Delta_{(i)}$ be a finite generating set for the R -module M_i . By Proposition 3.1, we know that the statement

$$\forall i = 1, 2, \dots, k : p_*(M_i) \text{ is a } \mathcal{K}\text{-module}$$

holds if and only if there exist \mathcal{K} -partitions $\pi_{(i)}$ of $\text{supp}(\Delta_{(i)})$, for $1 \leq i \leq k$, such that all components of p are constant on all parts of $\pi_{(i)}$. The latter condition is equivalent to saying $p \in \bigcap_{i=1}^k H(\pi_{(i)})^t$. So the requisite finite family of subgroups of \mathbb{Z}^{s*} is given by the family of intersections $\bigcap_{i=1}^k H(\pi_{(i)}) = H(\{\pi_{(i)} \mid 1 \leq i \leq k\})$, with the $\pi_{(i)}$ ranging independently over all \mathcal{K} -partitions of $\text{supp}(\Delta_{(i)})$. — The additional properties follow as in Proposition 3.1. \square

4. JUMP LOCI FOR THE RANK OF MATRICES

Let A be a matrix with entries in $R[\mathbb{Z}^s]$. We apply the results of the previous section to analyse the dependence of the rank of the matrix $p_*(A)$ from the homomorphism $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. First, we need to clarify what we mean by “rank”.

The \mathcal{K} -rank of a matrix. We denote by \mathcal{K} a hereditary set of ideals of R in the sense of Definition 1.3, with R an arbitrary commutative ring. Given any matrix A we write $|A|$ for the set of its entries. Now let A be specifically an $m \times n$ -matrix with entries in $R[\mathbb{Z}^s]$, and let z be a k -minor of A , that is, the determinant of a square sub-matrix of A of size k . (We remark here that determinants are defined in the usual fashion, *via* a sum indexed by the symmetric group or, equivalently, using LAPLACE expansion and induction on k , and that determinants have all the usual properties. See the discussion in §2 of [McC39].) Let z' be the minor of $p_*(A)$ corresponding to z , for some fixed $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. Then we have $p_*(z) = z'$, as $p_*: R[\mathbb{Z}^s] \longrightarrow R[\mathbb{Z}^t]$ is a ring homomorphism. We also have $p_*(|A|) = |p_*(A)|$.

Definition 4.1. We say that A has \mathcal{K} -rank 0 and write $\text{rank}_{\mathcal{K}}(A) = 0$ if $|A|$ is a \mathcal{K} -set (that is, if the ideal generated by the coefficients of the entries of A is a \mathcal{K} -ideal). Otherwise, the \mathcal{K} -rank of A is the maximal integer $k = \text{rank}_{\mathcal{K}}(A) > 0$ such that the set of k -minors of A is *not* a \mathcal{K} -set (that is, the ideal generated by the coefficients of the k -minors of A is *not* a \mathcal{K} -ideal).

The following Lemma sheds some light on this definition.

Lemma 4.2. *If the set of k -minors of A is a \mathcal{K} -set, then so is the set of $(k+1)$ -minors.*

Proof. By the familiar expansion formula of determinants, each minor of size $k+1$ is a linear combination of minors of size k , and the claim follows as \mathcal{K} is a hereditary property. \square

If R is a (commutative) *unital* ring, the \mathcal{K}_1 -rank coincides with the rank of matrices over $R[\mathbb{Z}^s]$ as considered by MCCOY [McC42, §2]; we will show this in Proposition 4.6 below. If R is a field, the \mathcal{K}_0 -rank coincides with the usual rank from linear algebra.

Proposition 4.3. *For each $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ we have the inequality*

$$\text{rank}_{\mathcal{K}}(p_*(A)) \leq \text{rank}_{\mathcal{K}}(A) .$$

Proof. Let $k = 1 + \text{rank}_{\mathcal{K}}(A)$. The set of $k \times k$ -minors of A is a \mathcal{K} -set, by definition of rank, hence so is its image under p_* by Lemma 1.4. But this image is precisely the set of $k \times k$ -minors of $p_*(A)$, whence $p_*(A)$ must have \mathcal{K} -rank strictly less than k . \square

The McCoy-rank of a matrix. Let S denote a commutative *unital* ring, and let A be an $m \times n$ -matrix with entries in S .

Definition 4.4. We say that the matrix A has MCCOY-rank 0, written $\text{rank}_{\text{MCCOY}}(A) = 0$, if there exists a non-zero element of S annihilating every entry of A . Otherwise, the MCCOY-rank of A is the maximal integer $k = \text{rank}_{\text{MCCOY}}(A)$ such that the set of k -minors of A is *not* annihilated by a non-zero element of S .

In complete analogy to Lemma 4.2 one can show that *if the set of k -minors of A is annihilated by a non-zero element of S , then so is the set of $(k+1)$ -minors*. The relevance of the MCCOY-rank is revealed by the following result:

Theorem 4.5 (McCOY [McC42, Theorem 1]). *The homogeneous system of m linear equations in n variables represented by the matrix A has a non-trivial solution over S if and only if $\text{rank}_{\text{McCOY}}(A) < n$. \square*

We are of course mainly interested in the special case $S = R[\mathbb{Z}^s]$ with R a commutative unital ring, in which case we also have the notion of \mathcal{K}_1 -rank at our disposal. Note that we have the inequality

$$\text{rank}_{\mathcal{K}_1}(A) \geq \text{rank}_{\text{McCOY}}(A) ;$$

indeed, if the set of coefficients of the k -minors of A is annihilated by a non-zero element y of R so that $\text{rank}_{\mathcal{K}_1}(A) < k$, then y also annihilates the k -minors themselves so that $\text{rank}_{\text{McCOY}}(A) < k$ as well.

Conversely, assume that the (finite) set $X \subseteq R[\mathbb{Z}^s]$ of k -minors of A is annihilated by a non-zero element $x \in R[\mathbb{Z}^s] \cong R[X_1^{\pm 1}, X_2^{\pm 1}, \dots, X_s^{\pm 1}]$. For every $\ell \in \mathbb{Z}$ the element $u = X_1^\ell X_2^\ell \cdots X_s^\ell$ is a unit; we choose $\ell \gg 0$ such that $X' = uX$ and $x' = ux$ lie in the monoid ring

$$P = R[\mathbb{N}^s] \cong R[X_1, X_2, \dots, X_s] .$$

The ideal $\langle X' \rangle \leq P$ generated by X' is annihilated by $x' \neq 0$. By a result of McCOY [McC57, Theorem] we can thus find a non-zero element $y \in R$ annihilating $X' = uX$. But then y also annihilates $u^{-1}X' = X$, and as y is an element of R this implies that y annihilates the coefficients of the elements of X individually. This yields the inequality $\text{rank}_{\mathcal{K}_1}(A) \leq \text{rank}_{\text{McCOY}}(A)$. — We have shown:

Proposition 4.6. *If R is a commutative unital ring, the two quantities $\text{rank}_{\mathcal{K}_1}(A)$ and $\text{rank}_{\text{McCOY}}(A)$ coincide. \square*

Jump loci. As before, we denote by \mathcal{K} a hereditary set of ideals of R in the sense of Definition 1.3. Let $q \geq 0$, and let A_1, A_2, \dots, A_k be matrices (of various sizes) over $R[\mathbb{Z}^s]$, with \mathcal{K} -ranks $r_i = \text{rank}_{\mathcal{K}}(A_i)$. We want to characterise those $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ such that

$$\forall i = 1, 2, \dots, k : \text{rank}_{\mathcal{K}}(p_*(A_i)) < \text{rank}_{\mathcal{K}}(A_i) - q . \quad (4.7)$$

Of course such p cannot exist if $r_i \leq q$ for some i . Otherwise, we see from Lemma 4.2 that (4.7) is true if and only if for all indices i , the image of the set of $(r_i - q)$ -minors of A_i under p_* is a \mathcal{K} -set. Writing M_i for the R -submodule of $R[\mathbb{Z}^s]$ generated by the $(r_i - q)$ -minors of A_i , this is equivalent to saying that the modules $p_*(M_i)$ are \mathcal{K} -modules. Now Corollary 3.4 applies. Note that, by definition of rank, the modules M_i are *not* \mathcal{K} -modules. We have shown:

Theorem 4.8. *Let A_1, A_2, \dots, A_k be matrices (of various sizes) over the ring $R[\mathbb{Z}^s]$, and let $q \geq 0$. There are a number $\ell \geq 0$ and direct summands G_1, G_2, \dots, G_ℓ of \mathbb{Z}^{s*} such that for $p \in H$,*

$$\begin{aligned} \forall i = 1, 2, \dots, k : \text{rank}_{\mathcal{K}}(p_*(A_i)) < \text{rank}_{\mathcal{K}}(A_i) - q \\ \iff p \in \bigcup_{j=1}^{\ell} G_j^t . \end{aligned}$$

If there is an index i such that $r_i \leq q$ then $\ell = 0$. Otherwise, the groups G_j are proper subgroups of \mathbb{Z}^{s*} so that $\bigcup_{j=1}^{\ell} G_j^t \neq \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$, and moreover $\ell > 0$ if the set of $(r_i - q)$ -minors of A_i is contained in the augmentation ideal I_s of $R[\mathbb{Z}^s]$ for all i . \square

5. \mathcal{K} -BETTI NUMBERS AND THEIR JUMP LOCI

Let C be a chain complex (possibly unbounded) consisting of finitely generated free based $R[\mathbb{Z}^s]$ -modules; more precisely, we suppose that $C_k = (R[\mathbb{Z}^s])^{r_k}$ for certain integers $r_k \geq 0$. We call r_k the rank of C_k ; if R is unital, the (commutative) ring $R[\mathbb{Z}^s]$ has IBN and thus the isomorphism type of C_k determines r_k uniquely.

Our differentials lower the degree, $d_k: C_k \longrightarrow C_{k-1}$, and we assume that each map d_k is given by multiplication by a matrix D_k with entries in $R[\mathbb{Z}^s]$. (For non-unital R this potentially restricts the set of allowed differentials.) We define the \mathcal{K} -rank of the homomorphism $d_k: C_k \longrightarrow C_{k-1}$, denoted $\text{rank}_{\mathcal{K}}(d_k)$, to be the \mathcal{K} -rank $\text{rank}_{\mathcal{K}}(D_k)$ of the matrix D_k , as defined previously.

Definition 5.1. The k th \mathcal{K} -BETTI number $b_k^{\mathcal{K}} = b_k^{\mathcal{K}}(C)$ of C is

$$b_k^{\mathcal{K}}(C) = r_k - \text{rank}_{\mathcal{K}}(d_k) - \text{rank}_{\mathcal{K}}(d_{k+1}) .$$

Given a homomorphism $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$, we define a new chain complex $p_*(C)$ by setting $p_*(C)_k = (R[\mathbb{Z}^t])^{r_k}$, equipped with differentials denoted $p_*(d_k)$ given by the matrices $p_*(D_k)$. In case R is locally unital (so that the multiplication map $R[\mathbb{Z}^s] \otimes_{R[\mathbb{Z}^s]} R[\mathbb{Z}^t] \longrightarrow R[\mathbb{Z}^t]$, $x \otimes y \mapsto p_*(x) \cdot y$ is an isomorphism) we have $p_*(C) = C \otimes_{R[\mathbb{Z}^s]} R[\mathbb{Z}^t]$. — In general we have $b_k^{\mathcal{K}}(p_*(C)) \geq b_k^{\mathcal{K}}(C)$ by Proposition 4.3, with strict inequality if and only if at least one of the strict inequalities $\text{rank}_{\mathcal{K}}(p_*(d_k)) < \text{rank}_{\mathcal{K}}(d_k)$ and $\text{rank}_{\mathcal{K}}(p_*(d_{k+1})) < \text{rank}_{\mathcal{K}}(d_{k+1})$ is satisfied.

More generally, given $q \geq 0$ we want to characterise those $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ such that

$$b_k^{\mathcal{K}}(p_*(C)) > b_k^{\mathcal{K}}(C) + q . \quad (5.2)$$

This happens if and only if applying p_* lowers the \mathcal{K} -rank of d_k by at least $q + 1 - j$, and lowers the \mathcal{K} -rank of d_{k+1} by at least j , for some j in the range $0 \leq j \leq q + 1$. Stated more formally:

Lemma 5.3. *The inequality (5.2) holds if and only if there exists a number j with $0 \leq j \leq q + 1$ such that*

$$\text{rank}_{\mathcal{K}}(p_*(d_k)) \leq \text{rank}_{\mathcal{K}}(d_k) - (q + 1 - j) \quad (5.4a)$$

and

$$\text{rank}_{\mathcal{K}}(p_*(d_{k+1})) \leq \text{rank}_{\mathcal{K}}(d_{k+1}) - j . \quad (5.4b)$$

Proof. If such j exists, then we have indeed

$$\begin{aligned} b_k^{\mathcal{K}}(p_*(C)) &= r_k - \text{rank}_{\mathcal{K}}(p_*(d_k)) - \text{rank}_{\mathcal{K}}(p_*(d_{k+1})) \\ &\geq r_k - (\text{rank}_{\mathcal{K}}(d_k) - (q + 1 - j)) - (\text{rank}_{\mathcal{K}}(d_{k+1}) - j) \\ &= r_k - \text{rank}_{\mathcal{K}}(d_k) - \text{rank}_{\mathcal{K}}(d_{k+1}) + q + 1 \\ &> b_k^{\mathcal{K}}(C) + q . \end{aligned}$$

For the converse, suppose that for each j at least one of the inequalities (5.4a) and (5.4b) is violated. Inequality (5.4b) holds for $j = 0$, by Proposition 4.3; let $m \leq q + 1$ be maximal such that (5.4b) is true for $0 \leq j \leq m$. We must have $m \leq q$; otherwise (5.4a) must be violated for $j = m = q + 1$, resulting in the inequality $\text{rank}_{\mathcal{K}}(p_*(d_k)) > \text{rank}_{\mathcal{K}}(d_k)$ which is known to be nonsense, by Proposition 4.3 again.

We have established that (5.4b) holds for $j = m \leq q$ but is violated for $j = m + 1$, that is, we know

$$\text{rank}_{\mathcal{K}}(p_*(d_{k+1})) \leq \text{rank}_{\mathcal{K}}(d_{k+1}) - m$$

and

$$\text{rank}_{\mathcal{K}}(p_*(d_{k+1})) > \text{rank}_{\mathcal{K}}(d_{k+1}) - (m + 1)$$

which yields the equality

$$\text{rank}_{\mathcal{K}}(p_*(d_{k+1})) = \text{rank}_{\mathcal{K}}(d_{k+1}) - m .$$

As (5.4b) holds for $j = m$ we know that (5.4a) must be false for $j = m$. This, together with the previous equality, provides the estimate

$$\begin{aligned} b_k^{\mathcal{K}}(p_*(C)) &= r_k - \text{rank}_{\mathcal{K}}(p_*(d_k)) - \text{rank}_{\mathcal{K}}(p_*(d_{k+1})) \\ &< r_k - (\text{rank}_{\mathcal{K}}(d_k) - (q + 1 - m)) - (\text{rank}_{\mathcal{K}}(d_{k+1}) - m) \\ &= r_k - \text{rank}_{\mathcal{K}}(d_k) - \text{rank}_{\mathcal{K}}(d_{k+1}) + q + 1 \\ &= b_k^{\mathcal{K}}(C) + q + 1 , \end{aligned}$$

whence $b_k^{\mathcal{K}}(p_*(C)) \leq b_k^{\mathcal{K}}(C) + q$ so that (5.2) does not hold, as required. \square

We can now apply Theorem 4.8 for each *fixed* j in the range $0 \leq j \leq q + 1$ to the two matrices D_k and D_{k+1} , yielding a family of direct summands of \mathbb{Z}^{s*} characterising for which $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ the inequalities (5.4a) and (5.4b) hold for our given choice of j . To characterise for which p inequality (5.2) holds, we allow j to vary and take the union of all the groups G_j occurring. Collecting the information then results in the following:

Theorem 5.5. *Let C be a (not necessarily bounded) chain complex of finitely generated based free $R[\mathbb{Z}^s]$ -modules, with differentials given by matrices over $R[\mathbb{Z}^s]$. Let $q \geq 0$ and $k \in \mathbb{Z}$. There are a number $\ell \geq 0$ and direct summands G_1, G_2, \dots, G_ℓ of \mathbb{Z}^{s*} such that for $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$,*

$$b_k^{\mathcal{K}}(p_*(C)) > b_k^{\mathcal{K}}(C) + q \quad \Longleftrightarrow \quad p \in \bigcup_{j=1}^{\ell} G_j^t . \quad \square$$

Empty jump locus. We finish the paper with a curious observation on the minors of differentials in certain chain complexes of free modules. We keep the notation from above: R is a commutative ring, \mathcal{K} a hereditary set of ideals, and C a (possibly unbounded) chain complex consisting of finitely generated based free $R[\mathbb{Z}^s]$ -modules. As before we denote the k th differential by d_k , and insist that d_k is given by multiplication by a matrix D_k . In Theorem 5.5 we consider $q = 0$ and a fixed $k \in \mathbb{Z}$, and assume that $\ell = 0$ so the jump locus is the empty set. This means that for all $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$ the induced complex $p_*(C) = C \otimes_{R[\mathbb{Z}^s]} R[\mathbb{Z}^t]$ has the same k th \mathcal{K} -BETTI number as C , that is, $b_k^{\mathcal{K}}(p_*(C)) = b_k^{\mathcal{K}}(C)$.

By definition of BETTI numbers the equality $b_k^{\mathcal{K}}(p_*(C)) = b_k^{\mathcal{K}}(C)$ necessitates that $\text{rank}_{\mathcal{K}}(d_i) = \text{rank}_{\mathcal{K}}(p_*(d_i))$ for $i = k, k+1$. Write $r = \text{rank}_{\mathcal{K}}(d_i)$. In case $r > 0$ we let $j \leq r$ be a positive integer. The set of j -minors of D_i is not a \mathcal{K} -set, by definition of the \mathcal{K} -rank and Lemma 4.2. It follows that the set of j -minors of $p_*(D_i)$ is not a \mathcal{K} -set either. (Indeed, if it was a \mathcal{K} -set then $\text{rank}_{\mathcal{K}}(p_*(C)) < j \leq r = \text{rank}_{\mathcal{K}}(d_i)$ so that $b_i^{\mathcal{K}}(p_*(C)) > b_i^{\mathcal{K}}(C)$ contrary to our hypothesis.) In particular, not all j -minors of D_i can be contained in the augmentation ideal of $R[\mathbb{Z}^s]$ as otherwise the j -minors of $0_*(D_i)$ would all be trivial, and would thus form a \mathcal{K} -set. We have shown:

Proposition 5.6. *Suppose that $b_k^{\mathcal{K}}(p_*(C)) = b_k^{\mathcal{K}}(C)$ for all $p \in \text{hom}(\mathbb{Z}^s, \mathbb{Z}^t)$. Let $i = k, k+1$. For every positive integer $j \leq \text{rank}_{\mathcal{K}}(d_i)$ at least one j -minor of D_i is not contained in the augmentation ideal of $R[\mathbb{Z}^s]$. \square*

The Proposition applies to the following special case: R a unital integral domain, $\mathcal{K} = \mathcal{K}_0$ and C a contractible complex. For then $p_*(C)$ is contractible as well since tensor products preserve homotopies. It follows that C and $p_*(C)$ are acyclic, for any p ; as R is an integral domain, the (usual) BETTI numbers, corresponding to the specific choice of \mathcal{K}_0 as hereditary set of ideals, can be computed as the rank (in the usual sense) of the homology modules, which are all trivial. That is, both C and $p_*(C)$ have vanishing k th BETTI number for all $k \in \mathbb{Z}$. As $\text{rank}_{\mathcal{K}_0}(d_k) > 0$ is equivalent to $d_k \neq 0$, we conclude:

Corollary 5.7. *Suppose that R is a unital integral domain, and that C is a contractible chain complex of finitely generated free $R[\mathbb{Z}^s]$ -modules. For every non-zero differential d_k of C , and every positive $j \leq \text{rank}(d_k)$, at least one j -minor of its representing matrix D_k is not contained in the augmentation ideal of $R[\mathbb{Z}^s]$. \square*

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